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The sexagesimal place-value system inside and outside texts

Christine Proust[*]

Abstract

This article proposes a contribution to the reflection on the invention of the sexagesimal place-value notation, a way of representing numbers which was at the basis of the so-called "Babylonian" mathematics (mathematical texts written in cuneiform script in the Ancient Near East from the 3rd to the 1st millennium BCE). The history of the sexagesimal place-value notation has already been the subject of numerous studies. It is proposed here to shed some new light on this question not only by examining texts, but also by taking into account the environment of the texts: what happens outside the text itself? What are the activities that accompany the act of writing?

Key-words: sexagesimal place-value notation; Mesopotamia; balance account; marginal number; calculation device.

El sistema de valor posicional sexagesimal dentro y fuera de los textos

Resumen

Este artículo propone una contribución a la reflexión sobre la invención de la notación del valor posicional sexagesimal, una forma de representar números que estaba en la base de las llamadas matemáticas "babilónicas" (textos matemáticos escritos en escritura cuneiforme en el Antiguo Cercano Oriente desde el III al I milenio a.e.c.). La historia de la notación del valor posicional sexagesimal ya

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ha sido objeto de numerosos estudios. Se propone aquí arrojar algo de luz sobre esta cuestión no solo examinando los textos, sino también teniendo en cuenta el entorno de los mismos: ¿qué sucede fuera del texto mismo? ¿Cuáles son las actividades que acompañan al acto de escribir?.
Palabras clave: notación sexagesimal de valor posicional; Mesopotamia; balance de cuentas; número marginal; dispositivo de cálculo.

Many cuneiform administrative and mathematical texts pose an enigma: they provide the results of a large number of calculations, often highly technical, but the practical implementation of these calculations has left very few traces, if any. One possible explanation, widely shared by current historians, for this apparent contradiction, would be that most calculations were performed on a material device used in parallel with the act of writing. A small group of administrative and mathematical tablets dated to the late 3rd millennium BCE provides new evidence on the existence and possible use of calculation devices. These tablets exhibit specific numerical notations written in the margins of the main text. This article analyses how these marginal numbers reflect the back and forth between text and device, and how they enlighten the process of the invention of the sexagesimal place-value notation. This reflection is based on research led in collaboration with Xiaoli Ouyang (Fudan University, Shanghai) in the framework of the European project *Sciences in the Ancient Worlds* - SAW (Ouyang and Proust forthcoming).

The Sexagesimal Place-Value Notation: a historical overview

The importance of the sexagesimal place-value notation in the history of mathematics cannot be underestimated (see Appendix I for a detailed description). This notation reflects highly sophisticated sexagesimal calculation techniques that were developed in the Ancient Near East from the third to the first millennium BCE. Specific to mathematics at its early stages, the sexagesimal place-value notation was acclimatized to astronomy at the end of the first millennium in Southern Mesopotamia, then passed on to Greek, Indian and Medieval Latin astronomers, to finally remain in use today in the measurement of time (and angles).

The science of sexagesimal calculation gave rise to spectacular mathematical elaborations during the Old-Babylonian period (ca. 2000-1600 BCE). The famous Plimpton 322 tablet is the best known example. The greatest development of cuneiform mathematics took place within the framework of the Old-Babylonian scribal schools, with, in particular, the development of an elaborate theory of quadratic problems as a kind of "geometric algebra" (Høyrup 2002a). The earliest texts showing a systematic use of the sexagesimal place-value notation are tables of reciprocals dated to the Ur III period (ca. 2100-2000 BCE). However, indirect traces of the use of diverse forms of the sexagesimal place-value system for calculation are attested earlier during the 3rd millennium, mainly in connection with the problems posed by the evaluation of surfaces (Powell 1972; Foster and Robson 2004; Proust 2020). As we see, from its origins, the history of the sexagesimal place-value notation is inseparable from that of calculation, and more precisely of calculation devices. The Ur III period probably offered favorable conditions for the intense mathematical activity which led to the development, the refinement, and the definitive fixing of the sexagesimal place-value notation, in the form that it would keep throughout the history of cuneiform mathematics until the end of the first millennium. Some more detailed information on both the assumptions concerning the use of a calculating device, and on the crucial period of Ur III, will be useful to the reader.

The hypothesis of the use of a calculation device

The hypothesis of the use of a calculation device, alongside the writing of texts, was put forward by historians as early as the 1950s. This hypothesis is based on lexical, archaeological, paleographical and mathematical evidence. Some lexical clues were pointed out by Oppenheim in 1959, including an expression he identified in lexical lists that could refer to a counting board. Subsequently other authors have uncovered other possible lexical traces[1]. The archaeological arguments are the core of Denise Schmandt-Besserat works. She popularized the thesis that the tokens found in many sites

[1] Oppenheim 1959; Lieberman 1980. The expression identified by Oppenheim is giš.ŠID.ma = *iṣ-ṣi mi-nu-ti*, given in Sumerian and translated in Akkadian in a list of wooden objects; this expression includes a determinative for wood (giš, Akkadian *iṣum*, tree) and a word which means 'calculation' or 'number' (šid, Akkadian *minutum*).

in the Ancient Near East are the ancestors of numerical signs written on the earliest clay tablets. This thesis remains widely accepted today[2]. It is worth noting the recent discovery at Ziyaret Tepe, a neo-Assyrian site in south-eastern Turkey, of many tokens associated with administrative texts. The archeologists in charge of the excavations think that these tokens were used as a calculation tool for administrative purposes. According to them, the old practices of calculation with tokens pre-dating writing would not have ceased with it, but would have continued in tandem with writing: "Prehistoric 'book keeping' continued long after invention of writing"[3]. Paleographic evidence comes from the observation of the shapes of the signs that were used to represent numbers. Following Schmandt-Besserat, many authors have pointed out the similarity between some of the clay tokens and the shape of archaic numerals, suggesting that numeral signs represent tokens. Later, the writing technology changed and the archaic curvilinear signs evolved into cuneiform numerical signs (see Fig. 1)[4]. The mathematical evidence is provided primarily by the analysis of errors in calculations[5].

On these grounds, hypotheses concerning the aspects and operation of presumed calculation devices have been suggested recently. For example, the calculation device may have consisted of parallel bands, each intended to receive oblong tokens for units (corresponding to the vertical wedges of cuneiform writing) and round tokens for tens (corresponding to the chevrons of cuneiform writing). Sexagesimal calculation, especially multiplication, is easy to perform with this device. Christopher Woods (2016) imagines that the device would be similar to an abacus (beads strung on rods). Other kinds of counting boards can also be imagined (see Fig. 2).

The Ur III period

The so-called Ur III period refers to the last Sumerian dynasty that ruled in Mesopotamia, ca. 2100-2000 BCE. This period has delivered more than 100,000 administrative texts, representing about 10 % of the total written texts exhumed by archaeologists and looters in the Ancient Near East. The

[2] Schmandt-Besserat 1977; Schmandt-Besserat 1981; Schmandt-Besserat 1996, Liverani 1983; Nissen, Damerow, and Englund 1993.

[3] MacGinnis et al. 2014.

[4] Nissen, Damerow, and Englund 1993.

[5] Powell 1976; Høyrup 2002b; Proust 2000. A synthesis of the evidence of the use of a calculation device can be found in Woods 2016.

rulers of this dynasty developed state control over the economic activities that was unparalleled in antiquity. In this context the laws, the writing norms and the metrological systems were revised and unified. The heart of the empire was located in southern Mesopotamia (Fig. 3).

Marginal numbers

Let us now go in search of the invention of the sexagesimal place-value notation and of a possible device associated with it, in the vast amount of administrative documentation produced by the bureaucrats of Ur III. My colleague Xiaoli Ouyang, Fudan University Shanghai, and I were intrigued by traces of numbers that are sometimes found in the margins of administrative tablets dated to the Ur III period and have been retrieved from the cities of Nippur, Umma and Girsu (see map in Fig. 3). These numerical graffiti appear on the edges of the tablets, or in boxes specially arranged for them. As these numbers are noted outside the main text, we have called them "marginal numbers". Fig. 4 shows an example of a box containing marginal numbers (on the reverse side of the tablet).

To understand how marginal numbers give us clues about calculation processes, let's take one example among the many analyzed in Ouyang Proust forthcoming. The tablet YBC 4179, from Umma and dated Ur III, contains a 12-year balance account dealing with quantities of grain (see copy Fig. 5).

The text records the tracking over several years of transactions between the governor of Umma province and one of his suppliers, an administrator named Lu-Ninšubur. Each year, the supplier Lu-Ninšubur received from the governor an advance of grain for beer-making. The supplier Lu-Ninšubur spent this grain on behalf of the governor. The final balance is a "surplus", which means that at the end of the 12 years the supplier Lu-Ninšubur spent more than he received and the Governor owes him a debt[6].

Two marginal spaces containing numbers appear on this tablet. One is located on the upper edge (see Fig. 6). We see 7 "aš" signs followed by a number apparently in sexagesimal place-value notation[7]. This inscription can be represented as follows

[6]An unusual situation: in most of the Ur III balance accounts, the final balance is a "deficit".

[7]"aš" is the Sumerian name (and probable pronunciation) of the number "one" noted with a horizontal wedge. "diš" is the Sumerian name (and probable pronunciation) of the number "one" noted with a vertical wedge.

7(aš) 1:46:30

The second marginal number is located in a box in the 2nd column on the reverse (column *ii*, at the right), after the balance and before the name of the supplier and the dates (see Fig. 6). We see 5 "aš" signs followed by a space, then a number apparently in SPVN:

5(aš) /space/ 1:30

Note that the writing of numbers is not standardized in the main text, and is standardized in the margins (see appendix III). This is a general phenomenon, which can be observed in all the other examples of marginal numbers.

Let us first look at the marginal inscription on the upper edge. The comparison of the marginal number 7(aš) 1:46:30 with the quantities recorded in the account shows that it corresponds to the quantity of grain 7 *gur* 1 *barig* 4 *ban* 6 1/2 *sila*, i.e. the total of grain advances for beer production over the first ten years of the account (see appendix II for explanations on the metrological system used for capacities of grain). This quantity appears in the main text on the reverse, column *i*, line 5.

Indeed, this quantity is composed of 7 *gur*, represented in the upper edge with seven "aš", and 1 *barig* 4 *ban* 6 1/2 *sila*, represented on the upper edge in sexagesimal place-value notation by the number 1:46:30. As we see, the numerical system used in the margin is not the same as in the main text, and the measurement units and fractions have disappeared from the margin. The arithmetic structure of the quantity can be represented in the following tables.

The structure of the metrological notation in the main text can be represented as follows:

gur	barig	ban	sila
7(aš)	1	4	6 1/2

This structure explains the form of the corresponding marginal number (knowing that 10 *sila* is 1 *ban*; 6 *ban* is 1 *barig*; 5 *barig* is 1 *gur* – see appendix II):

gur	barig-ban-sila
7(aš)	1 : 46 : 30

In the marginal number, the number of gur is recognizable by the shape of the signs (horizontal wedges "aš"). Therefore, it is not a strict place-value system. In addition, the factor between the units *barig* and *gur* is 5 (not 60) (see appendix II), so the system is not strictly sexagesimal either.

The second marginal number is located in a box in the middle of the last column (col. *ii* on the reverse). This marginal number exhibits the same characteristics as that noted on the upper edge: the numerical system is partially sexagesimal and partially place-valued. This marginal number is the result of the difference between the total quantity of grain advanced by the governor (60+48 *gur* 4 *barig* 5 *ban*, line 7 of col. *i* on the reverse), and the total quantity of grain expended by the supplier (60+53 *gur* 4 *barig* 5 *ban* 1 1/2 *sila*, line 1 of col. *ii* on the reverse).

Why is the same quantity represented twice, once in the main text using usual metrological notation, and once in the margins (upper edge or box), using an unusual numerical notation partially sexagesimal, partially place-valued, and without mention of the measurement units? As the notations of the same quantity in the main text and in the margins are different, the function of these notations should be different. In the margins, the reduction of the notation to pure numbers, without measurement units, in a system close to a sexagesimal place-value notation suggests that the function of marginal numbers has something to do with calculation. These calculations are the addition of ten quantities of grain in the first case, and the subtraction of the total amount of grain advanced by the governor from the total amount of grain expended by the supplier in the second case. The formatting of quantities in the margin seems to be related to the requirements of performing an addition of many terms, and a subtraction.

For example, the marginal number in the box (second case) would reflect the following calculation:
The quantity 60+53 *gur* 4 *barig* 5 *ban* 1 1/2 *sila* would be transformed into marginal number:

gur	barig-ban-sila
1:53 (aš)	4: 51: 30

The quantity 60+48 gur 4 barig 5 ban would be transformed into marginal number too:

gur	barig-ban-sila
1:48 (aš)	4: 50

The marginal numbers would be transferred on a device to perform the subtraction:

1: 53	4: 51: 30
1: 48 (aš)	4: 50

The difference is:

5	1:30

The marginal number 5 (aš) 1: 30 corresponds to the quantity 5 gur $1\frac{1}{2}$ sila.

In other texts, detailed in (Ouyang and Proust forthcoming), the marginal numbers are noted in a completed form of sexagesimal place-value notation and they relate to multiplications by a rate. The analysis of the available examples of marginal numbers shows that the formatting of the quantities in the margins depends on the operations to be performed: partially place-valued for additions and subtractions, pure sexagesimal place-value notation for multiplications and divisions.

To sum up, the features of the marginal numbers are different from those of the metrological notations in the main text. Marginal numbers are noted in a dedicated space on the tablet (special boxes or edges). The numerical system adopted in the margins is different from that in the main texts, and moreover this system depends on the operations to be performed (partial sexagesimal place-value notations for additions and subtraction; pure sexagesimal place-value notation for multiplication by a rate). The shapes of the signs are normalized, unlike in the main text (see appendix III). The writing is cursive, superficial and easy to erase (the signs are light, not deep in the clay). Numerous traces of erasures are present in the margins.

These observations converge to the hypothesis that marginal numbers are ephemeral intermediate stages between writing and a calculation device.

Main text	Transformation	Margin	Transfer	Calculation device
Quantities	⇒	Marginal numbers	⇒	Inputs
Quantities	⇐	Marginal numbers	⇐	Outputs

Table 1: From writing to a calculation device and vice versa (after Ouyang Proust forthcoming)

Table 1 proposes a representation of the possible process of calculation. The quantities to be added (or subtracted, or multiplied by a rate) are noted in the main text with standard metrological notations. The problem, for the users of administrative tablets, is that these notations are not always suitable for calculation. These metrological notations are thus transformed into numbers suitable for calculation in the margins and then transferred to a calculation device. The operations are performed with this device (or perhaps, in simple cases, mentally), then the output is transferred back to a margin after erasing the previous marginal numbers, and finally transformed back into metrological notations in the main text.

The margins would play the role of a scratchpad, where quantities are formatted for calculation and then erased, which explains the numerous traces of erasures in these spaces. Note that blank spaces containing erasures are common in administrative texts (see example in Fig. 7), while examples of marginal numbers are relatively rare in comparison with the enormous quantity of known administrative texts[8]. Embedding scratchpads in administrative texts seems to have been a common practice. However, the graffiti themselves were most often erased upon completion of the text, which explains the frequency of the blank spaces and the rarity of marginal numbers scribbled on it.

Text and context

The relationship between text and instrument suggested with the example of the YBC account allows one to imagine the complex sequence of actions taken by an administrator who draws up a balance sheet of transactions over 12 years. First, the scribe must have each of the 12 annual balance sheets available to him to transfer the balance (governor's advance and supplier's expenditure) to his multi-year account. One can imagine him working on writing his multi-year balance sheet in the middle of a multitude of auxiliary tablets. Once the data is collected and written on his tablet, he proceeds to add up the receipts on the one hand and the disbursements on the other. For this, the scribe transforms the metrological data into more practical notations and transfers them to the margins. Doing so, the scribe must abandon the linear flow of his text to jump to the margins. The scribe then turns to his instrument to perform the calculations. The marginal space is cleaned up before transferring the result of the calculation. This new marginal number is transformed back into metrological notation and the scribe can resume the linear flow of his text.

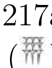
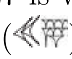
[8]Xiaoli Ouyang and I counted some twenty texts containing marginal numbers similar to those described in this article (i.e. written in a partially or fully positional system) – see Ouyang and Proust forthcoming Appendix 5.III.

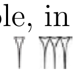
To conclude, the marginal numbers are testimonies of a mathematical activity that connects arithmetical operations to numbers. In administrative texts implying multiplications by a rate (for example, evaluating the price in silver of goods), marginal numbers show how the multiplications and divisions are connected to the sexagesimal place value notation. Together with the fact that the earliest known mathematical texts using sexagesimal place-value notation are reciprocal tables, which were employed in multiplications and divisions, the evidence is strong to sustain that such notation emerged in response to needs of calculation. The observations and hypotheses developed in this article are part of a larger reflection on the links between calculation and numbers in the ancient worlds that has been developed in the SAW project (Chemla, Keller, and Proust forthcoming). It is a new chapter in the history of numbers that is opened in the introduction of this book (Chemla forthcoming).

Appendices

Appendix I: the Sexagesimal Place-Value Notation (SPVN)

The sexagesimal place-value notation is a numerical system that was used in mathematical cuneiform texts dated to the Ur III period onward. As indicated by its name, the base is sixty and the system is place-valued (the value of the signs depends on its position in the number). The system can be easily understood looking at a text such as this multiplication table by 9 (Fig. 8).

The numbers from 1 to 59 are written by repeating ones (wedges ) and tens (chevrons ) as many times as necessary. For example, in line 1 of the tablet HS 217a (Fig. 8), the number 9 is written by repeating the wedge nine times (); in line 3, the number 27 is written by repeating the chevron two times and the wedge seven times ().

The numbers beyond 59 are written in sexagesimal place-value notation. For example, in line 7 of the tablet HS 217a (Fig. 8), the product 9 times 7 is written  (transliteration 1:3), a two-place number where the wedge in the left-place is worth 60 times more than each wedge in the right-place.

The notation does not indicate the position of the units in the numbers. For example, the number  in line 3 is noted in the same way as the product 9 times 20 in line 20 (reverse). **The notation is floating**[9].

Appendix II: metrology

To understand the notations in the texts quoted, some information on metrological systems will be useful, particularly on units of capacity. The metrological system for capacity is as follows (the units are enumerated in increasing order from right to left, and the factors which define each unit from the previous one is noted in between):

gur	←x5	*barig*	←x6	*ban*	←x10	*sila*
≈300 liters		≈60 liters		≈10 liters		≈1 liters

Note that **the factors that define one unit from the others** are 10, 6, and 5, so the structure is not completely sexagesimal.

As for the numerical values, they are expressed differently for small and large units.

To count small units, namely *sila* (unit of capacity), the following numerical signs are used:

- the number 1 is represented by a vertical wedge, read "diš"
- the number 10 is represented by a chevron, read "u"
- the fractions 1/3, 1/2, 2/3, 5/6 are represented by special signs.

To count large units, namely *gur* (the largest unit of capacity) the following numerical signs are used:

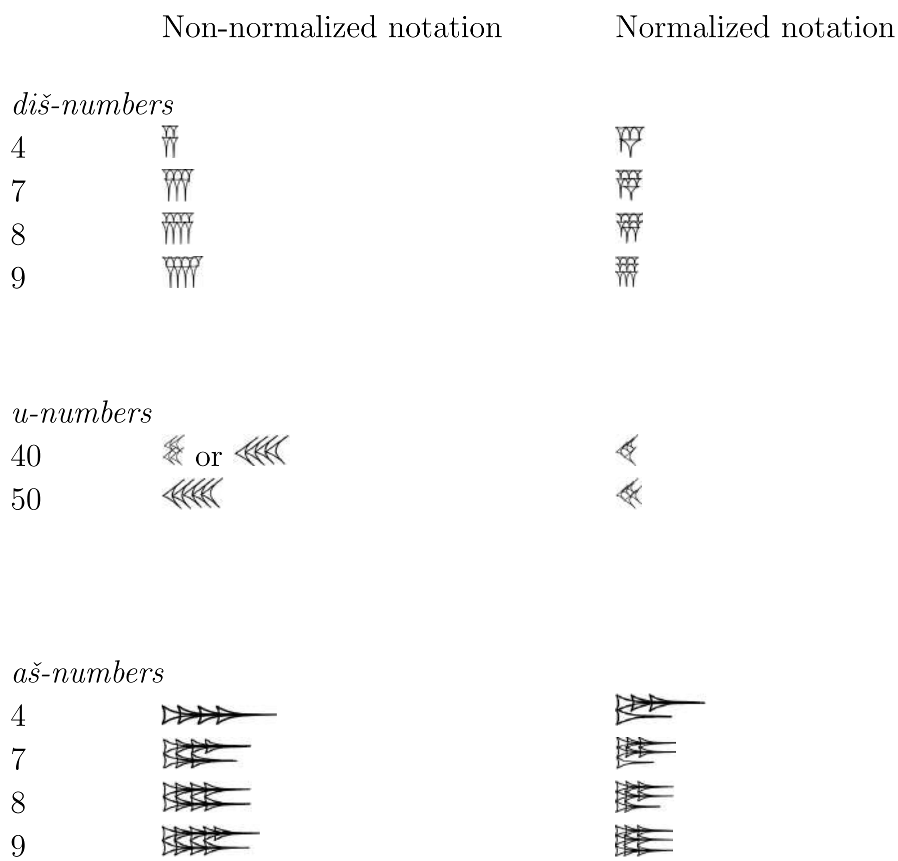
- the number 1 is represented by a horizontal wedge, read "aš",
- the number 10 is represented by a chevron, read "u"
- the number 60 is represented by a vertical wedge, read "geš" which allows 60 to be distinguished from 1.

[9]For more details on the floating notation and implications on the interpretation of mathematical texts, see Proust 2013.

The signs for one and ten are repeated as many times as necessary to represent the desired quantity. Note that the arrangements of the wedges and chevrons are not the same in the main text (non-normalized) and in the margins (normalized).

Appendix III: paleography

In the majority of cases, the cuneiform signs for numbers are obtained by writing, as many times as necessary, the elements that compose the number (vertical wedges, pronounced "diš", for ones, chevrons, pronounced "u", for tens, or horizontal wedge, pronounced "aš", also for ones). The way in which these elements are arranged vary according to their use: in the main text, the elements are arranged into one or two rows ("non-normalized" notation); in the margin, the elements are arranged into one, two or three rows, each row including three elements if possible ("normalized" notation) -see Fig. 9.



List of figures

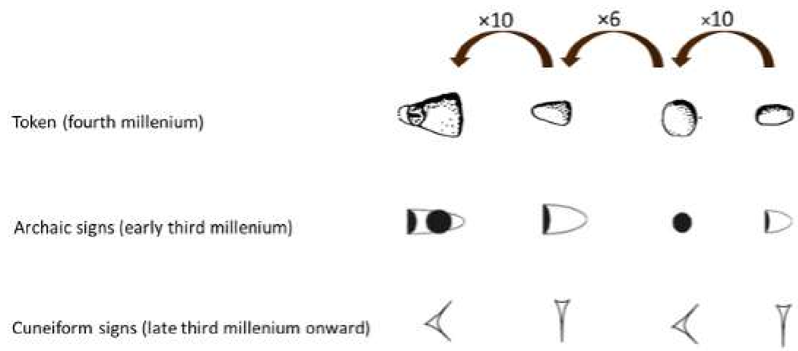


Figure 1: Token (IVth millennium), archaic numerals (early IIIrd millennium) and cuneiform signs (late IInd millennium onward) representing the numbers one, ten, sixty and six hundred (Schmandt-Besserat 1996, Nissen, Damerow, and Englund 1993).

Figure 2: Two reconstructions of possible calculating devices, which were experimented in the framework of the SAW project (a counting board and an abacus built by Baptiste Mélès).

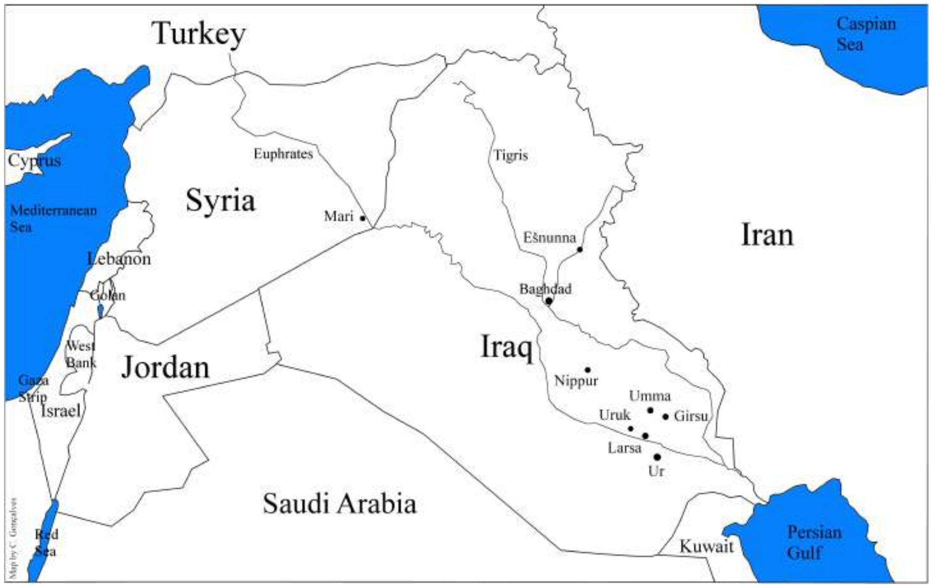


Figure 3: Map of Mesopotamia, with present-day national borders and ancient cities mentioned in this text

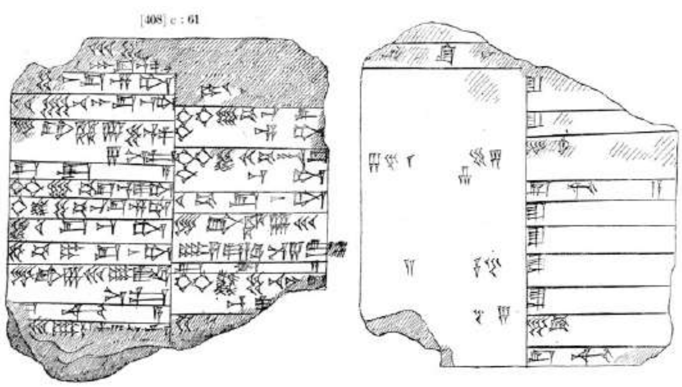


Figure 4: An administrative tablet preserved at the Louvre Museum, Paris, under the number AO 27307; the tablet comes from Girsu, is dated Ur III, and contains an evaluation of the amount of seed needed to sow different plots. Numerical graffiti appear in the bottom part of the left column of the reverse. (Copy Thureau-Dangin 1903 n° 61 – see also the photo at https://cdli.ucla.edu/P128561)

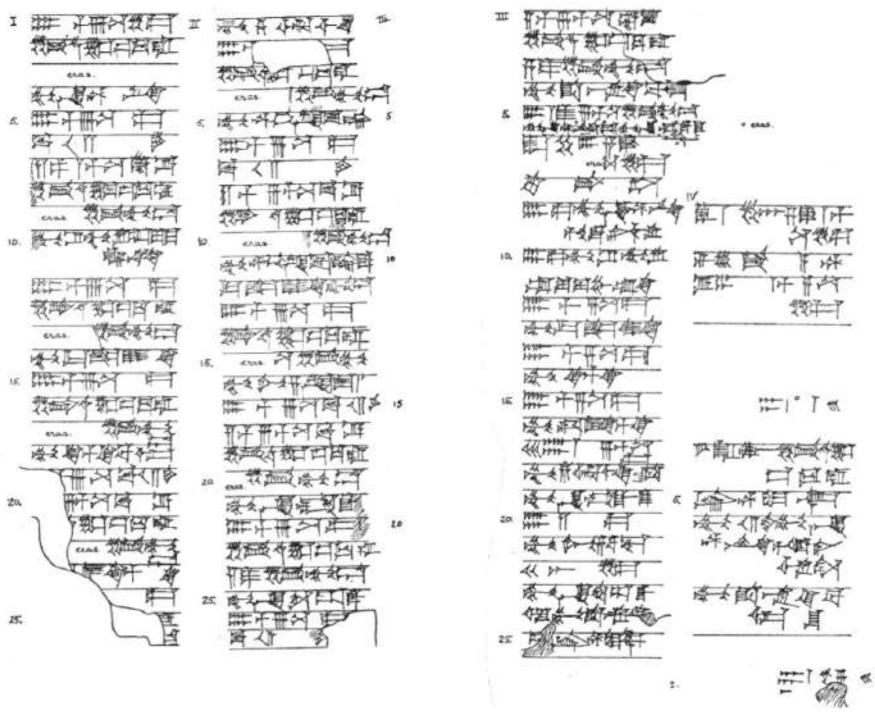


Figure 5: An administrative tablet preserved at Yale University under the number YBC 4179. Two boxes containing numeral graffiti are visible: one in the middle of the right column of the reverse, the second on the upper edge (copied at the bottom of the autograph). (Copy Ellis 1970 - see also the photo at https://cdli.ucla.edu/P111807)

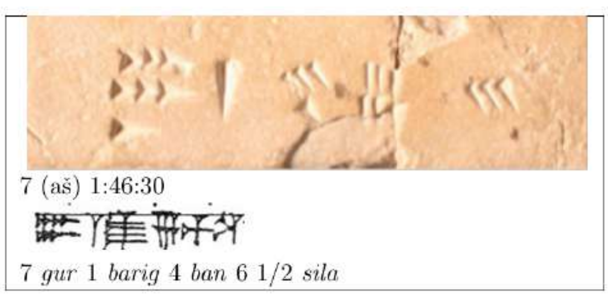


Figure 6: The marginal number noted on the upper edge of YBC 4179 and the corresponding quantity noted in the main text (reverse, column i, line 5).

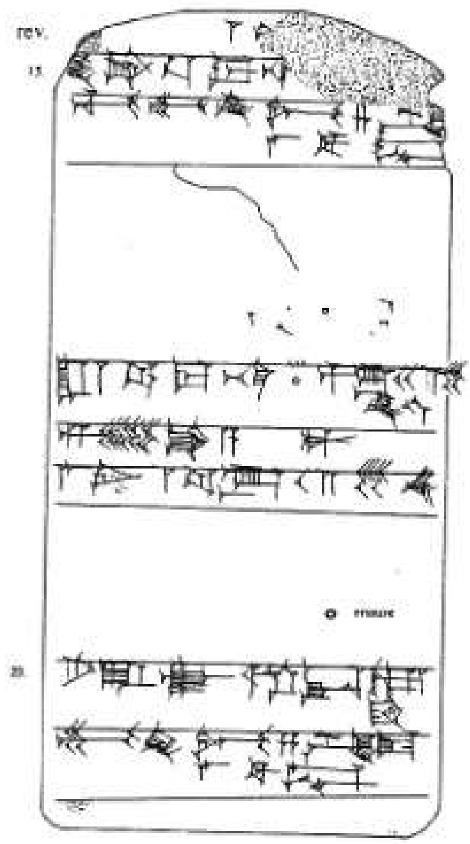


Figure 7: Two blank spaces containing erasures on the reverse of administrative tablet MLC 1980 from Umma, dated Ur III (copy Snell 1982)

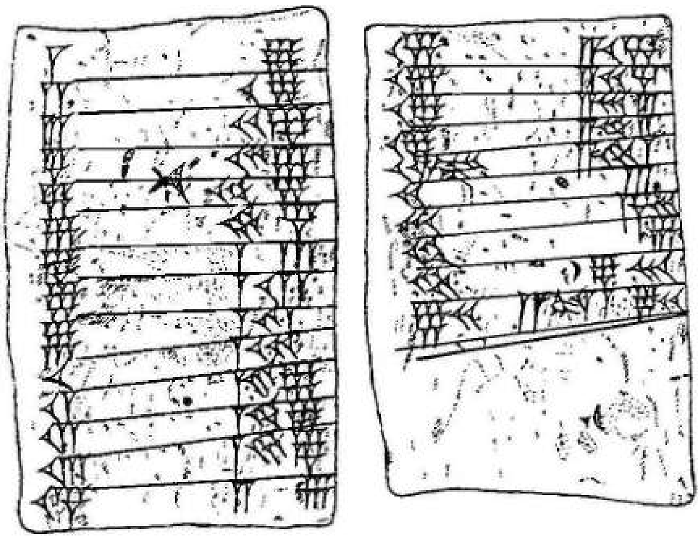


Figure 8: HS 217a preserved at the Jena University (copy Hilprecht 1906). School tablet from Nippur dated to the Old Babylonian period – see photo here (`https://cdli.ucla.edu/P254585`)

Abbreviations

AO Antiquités Orientales, the Louvre Museum
BCE Before the Common Era
SAW Sciences in the Ancient Worlds. European project 2011-2016 led by Karine Chemla, in collaboration with Agathe Keller and Christine Proust, European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement n. 269804.
SPVN sexagesimal Place-Value Notation
YBC Yale Babylonian Collection

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